ECON4930 Solution to Seminar exercise 1, Spring 2011

1. Consider the following social planning problem for utilisation of water over time within a hydro generating system given the past investment in generation capacity, transmission network, etc.

$$\max \sum_{t=1}^{T} \int_{z=0}^{e_t^H} p_t(z)\,dz$$
subject to
$$R_t \le R_{t-1} + w_t - e_t^H$$
$$R_t \le \bar{R}$$
$$R_t, e_t^H \ge 0$$
$$T, w_t, R_o, \bar{R} \text{ given, } R_T \text{ free, } t = 1,..,T$$

where

$$e_t^H = \text{electricity production during period } t \text{ (kWh)}$$
$$p_t(e_t^H) = \text{demand function}$$
$$R_t = \text{reservoir level at end of period } t \text{ (kWh)}$$
$$w_t = \text{inflow during period } t \text{ (kWh)}$$
$$\bar{R} = \text{reservoir constraint (kWh)}$$

a) Explain the relations in the planning problem, starting with the objective function, and comment upon underlying assumptions.

Partial model, $U_t'(e_t^H) = p_t(e_t^H), p_t'(e_t^H) < 0$

Water accumulation,

$$0 \le R_t \le \bar{R}$$
$$R_t \le R_{t-1} + w_t - e_t^H$$

When overflow, then $R_t < R_{t-1} + w_t - e_t^H$, $R_t = \bar{R}$

b) Derive the first-order conditions for the problem. Make economic interpretations introducing reasonable assumptions.

$$\frac{\partial L}{\partial e_t^H} = p_t(e_t^H) - \lambda_t \le 0 \ (= 0 \text{ for } e_t^H > 0)$$

$$\frac{\partial L}{\partial R_t} = -\lambda_t + \lambda_{t+1} - \gamma_t \le 0 \ (= 0 \text{ for } R_t > 0)$$

$$(\text{constraint for period } t+1 : -\lambda_{t+1}(R_{t+1} - R_t - w_{t+} + e_{t+1}^H)$$

$$\lambda_t \ge 0 (= 0 \text{ for } R_t < R_{t-1} + w_t - e_t^H)$$

$$\gamma_t \ge 0 (= 0 \text{ for } R_t < \bar{R}) , \qquad t = 1,..,T$$

Reasonable assumption: $e_t^H > 0$ for all t. Implication: price equal to water value for all t.

c) Discuss qualitative aspects of possible optimal solutions regarding electricity prices, focussing on possibilities for price changes. In your discussion try to make use of the principle of backward induction, i.e. starting with period T (hint: find the optimal solution for period T, then move to period T-1, etc. Try to illustrate your discussion using bathtub diagrams, and make sure that the diagrams conform with the first-order conditions.

Price changes when hitting reservoir constraints 0 or $\bar{R}$.

Backward induction: Optimal solution for period T, first-order conditions

$$p_T(e_T^H) = \lambda_T \ (e_T^H > 0)$$

Reasonable assumption: $Max_t \{p_t(Max\, e_t^H)\} = Max_t \{p_t(\bar{R} + w_t)\} > 0 \Rightarrow p_T > 0$.

$$-\lambda_T + \lambda_{T+1} - \gamma_T \le 0 \ (= 0 \text{ for } R_T > 0)$$

Due to the terminal condition of free reservoir level for end of period T it will be optimal to empty the reservoir in period T, leading to $-\lambda_T \le 0$, becase $\lambda_{T+1} = 0$ and $\gamma_T = 0$ since $R_T = 0$. Due to no-satiation assumption we then have $\lambda_T = p_T > 0$.

Possible price changes are caused by the reservoir level hitting the constraints 0 or $\bar{R}$. We can use bathtub diagrams for these situations. Starting from t=T and moving backwards to T-1, etc. we will assume that the reservoir for a number of periods are in between empty and full, $0 < R_t < \bar{R}$, Then it follows that the water values and prices must be equal for all such consecutive periods.

Let us assume that for period t+1 the reservoir is emptied. From the first-order condition for t+1 we then have

$-\lambda_{t+1} + \lambda_{t+2} - \gamma_{t+1} \leq 0 \; (R_{t+1} = 0 \Rightarrow \gamma_{t+1} = 0) \Rightarrow \lambda_{t+1} \geq \lambda_{t+2}$. From the previous assumption about develoment backwards from T we have that $\lambda_{t+2} = p_T$. The typical situation will then be that $\lambda_{t+1} > \lambda_{t+2} = p_T$ in order for using up all water in t+1 to be optimal.

Assume that we have $0 < R_t < \overline{R}$ for some consecutive periods before t+1 going backwards. All prices for these periods are constant and equal to λ_{t+1}. When we come to period s there is a threat of overflow

$-\lambda_s + \lambda_{s+1} - \gamma_s = 0 \; (R_t = \overline{R} > 0 \Rightarrow \gamma_s \geq 0)$. The typical situation is that $\gamma_s > \Rightarrow \lambda_s = \lambda_{s+1} - \gamma_s \Rightarrow \lambda_s < \lambda_{s+1}$. The price of period s will be lower than the price of period s+1. If for the remaining periods backwards to period1 we have the resevoir level in between enpty and full, then the prices will be equal and equal to $\lambda_s = p_s = p_{t+1}$.

d) Introduce a constraint on the generation of electricity $e_t^H \leq \overline{e}^H$. Explain the interpreation of the constraint, and study the consequence for the possibility of price changes of the new optimal solution (hint: introduce the constraint in the problem in question a), and expand the Lagrangian function correspondingly). Try to illustrate using a bathtub diagram.

The new constraint in the Lagrangian for the problem: $-\rho_t(e_t^H - \overline{e}^H), t = 1,..,T$.

The new first-order condition for water in period t is:

$$p_t(e_t^H) - \lambda_t - \rho_t \leq 0 \; (= 0 \text{ for } e_t^H > 0)$$
$$\rho_t \geq 0 (= 0 \; for \, e_t^H < \overline{e}^H)$$

If the production constraint is binding, then the social price is (typically) greater than the water value, $p_t(e_t^H) = \lambda_t + \rho_t$.